Int. J. Heat Mass Transfer. Vol. 10, pp. 207–217. Pergamon Press 1967. Printed in Great Britain

DIFFUSION ACROSS A LAMINAR COMPRESSIBLE MIXING ZONE

GERALD SCHUBERT†

Bell Telephone Laboratories, Incorporated, Whippany, New Jersey

(*Received* 6 *September* 1965 *and in revised form* 30 *May* 1966)

Abstract—The problem considered in this paper is the laminar mixing of two parallel streams of compressible fluid mixtures. The streams are two component compressible fluids, e.g. air and some contaminant. The primary interest is to describe the mixing process and its effect on the concentration of each species composing the fluid. The concentration of one of the components of the binary mixture is assumed to be small, and a perturbation solution, with the concentration as the small parameter, is initiated. The zero-th approximation is carried out in detail. The Schmidt number and the Prandtl number are arbitrary constants, the viscosity is assumed to vary linearly with temperature, and the two components of the fluid mixture individually obey a perfect gas equation of state. An analytical solution to the initial approximation is obtained by means of rapidly convergent series expansions. Only the Schmidt number and the Prandtl number influence the computation of fundamental solutions.

NOMENCLATURE

x, y and z, Cartesian coordinates;
ρ, mass density;
u, x component of velocity;
v, y component of velocity;
c, mass fraction;
μ, dynamic viscosity;
ν, kinematic viscosity;
D_{12}, binary diffusion coefficient;
Sc, Schmidt number, ν/D_{12};
c_p, specific heat at constant pressure;
T, temperature;
Pr, Prandtl number, $(\mu/k)\{c_{p_2} + c(c_{p_1} - c_{p_2})\}$;
k, thermal conductivity;
R, gas constant;
p, pressure;
h, enthalpy;
Y, transformed y coordinate (see equations 9);
ψ, stream function (see equations 9);
η, similarity parameter, $Y^*(2x^*)^{-\frac{1}{2}}$;
f, function defined by $\psi^* = \sqrt{(2x^*)}f(\eta)$;
b, scale factor in the solution of f_o (see equation 21);
ξ, modified similarity parameter, $\xi = b\eta$;
$\bar{f}_o$, function defined by $\bar{f}_o = (1/b)f_o$;
$\bar{c}_o$, function directly proportional to c_o^* (see equation 29);
$\bar{T}_o$, function defined by $(T^+/T^-)\,T_o^* - 1 = \bar{T}_o$;
E, parameter in the solution of the temperature equation given by $(u^{+2}/c_{p_2}T^+)(T^+/T^-)\,b^4$;
$\bar{T}_{op}$, a particular solution of equation (37);
$\bar{T}_{oh}$, a homogeneous solution of equation (37);
K, a constant in the solution of $\bar{T}_o$ (see equation 39).

Subscripts

p, refers to the particular solution of a differential equation;
h, refers to the homogeneous solution of a differential equation;

† Presently NAS–NRC Postdoctoral Fellow, Department of Applied Mathematics and Theoretical Physics, Cambridge University, Cambridge, England.

1 and 2, refer to the two components of the fluid mixture;
o, refers to the initial approximation.

Superscripts
$+$, refers to conditions far from the mixing region in the upper stream;
$-$, refers to conditions far from the mixing region in the lower stream;
$*$, refers to a dimensionless variable.

INTRODUCTION

SEVERAL authors have considered the problem of the laminar boundary layer between parallel streams. Chapman [1] investigated the mixing of a high-velocity stream with a region of fluid at rest. His analysis assumed that the Prandtl number was unity and the method of solution involved repeated quadratures. Lessen [2], in a paper on the stability of the flow of a stream of incompressible fluid over a layer of the same fluid at rest, considered analytical methods only far enough to reduce the problem to a routine numerical solution. The more general problem, when the two fluids are of different densities and viscosities, was solved by Lock [3] who also used analytical methods only to facilitate a numerical integration.

The mixing of two semi-infinite incompressible streams has also been studied by Görtler [4] and Pai [5]. Görtler's method assumes that the streams have nearly identical velocities so that a series expansion in powers of a small parameter, which is the dimensionless velocity difference between the streams, is used. Crane [6] has used a double series of powers of two parameters and the method of Görtler to deal with the compressible mixing problem.

The problem considered in this paper is the two-dimensional laminar mixing of two parallel streams of compressible fluid mixtures. Each stream is a two-component compressible fluid, e.g. air and some nonreacting contaminant. Far from the mixing region the properties of each stream, such as velocity, temperature, density, contaminant concentration, etc., are constant.

The conservation equations describing the mixing process must be solved subject to the conditions that the velocity, temperature, density and concentration both far above and below the mixing region must tend to be prescribed constant values.

Two distinct physical situations arise which lead to perturbation solutions of the general mixing problems; namely when the streams have nearly equal contaminant concentration and when the lower stream is at rest and uncontaminated, while in the upper stream, the concentration of contaminant is very small. The bulk of this paper will deal with the latter case, for which the self-similar solution to the initial approximation, (for $\mu \sim T$, $Pr =$ constant and $Sc =$ constant), is calculated by a method of power-series expansion and analytic continuation. This method is closely related to the method used by Blasius [7] in his famous paper on the solution of the boundary layer adjacent to a flat plate.

PROBLEM FORMULATION

Consider the two-dimensional mixing of two streams of fluid. Far from the mixing region the upper fluid has velocity u^+, temperature T^+, density ρ^+, viscosity μ^+ and contaminant concentration c^+. The velocity u^+ is directed along the positive x-axis and the positive y-axis points into the upper fluid. The lower fluid has characteristics u^-, T^-, ρ^-, μ^- and c^- far from the mixing region. The problem is depicted in Fig. 1.

The equations of conservation of mass, momentum and energy for a multicomponent fluid are derived in Truesdell and Toupin [8] from a continuum point of view and in the text by Hirschfelder *et al.* [9] from a microscopic viewpoint. It is assumed here that the boundary layer approximation to these equations is applicable. A boundary layer analysis is justified far downstream from the onset of the mixing region. The appropriate equations are (see Fay and Riddell [10])

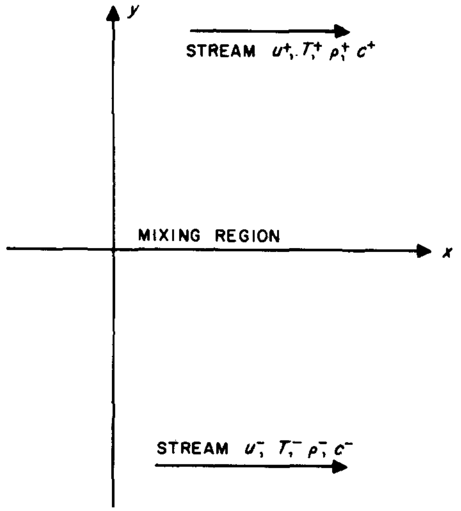

FIG. 1. The geometry of the mixing problem.

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

$$\rho q \frac{\partial c}{\partial x} + \rho v \frac{\partial c}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Sc}\frac{\partial c}{\partial y}\right) \tag{2}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{3}$$

$$\begin{aligned} c_{p_2}\left(1 + \frac{c_{p_1} - c_{p_2}}{c_{p_2}}c\right)\left(\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y}\right) &= \mu\left(\frac{\partial u}{\partial y}\right)^2 + c_{p_2}\left(1 + \frac{c_{p_1} - c_{p_2}}{c_{p_2}}c\right)\frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial T}{\partial y}\right) \\ &\quad + \mu c_{p_2}\left(\frac{c_{p_1} - c_{p_2}}{c_{p_2}}\right)\frac{\partial T}{\partial y}\frac{\partial c}{\partial y}\left(\frac{1}{Pr} + \frac{1}{Sc}\right) \end{aligned} \tag{4}$$

and the equation of state is assumed to be

$$p = \text{constant} = \rho T R_2\left(1 + \frac{R_1 - R_2}{R_2}c\right) \tag{5}$$

i.e. each component of the mixture is a perfect gas with constant specific heat.

c is understood to be the mass concentration of species 1. Species 2 is usually considered to be air and species 1 as contaminant. This system of equations will be investigated under the assumption that Pr and Sc are constants.

The boundary conditions are

$$\lim_{y\to\infty}\frac{c}{c^+} = 1, \quad \lim_{y\to\infty}\frac{u}{u^+} = 1, \quad \lim_{y\to\infty}\frac{T}{T^+} = 1$$

$$\lim_{y\to-\infty}\frac{c}{c^-} = 1, \quad \lim_{y\to-\infty}\frac{u}{u^-} = 1$$

and

$$\lim_{y\to-\infty}\frac{T}{T^-} = 1. \tag{6}$$

METHOD OF SOLUTION

The solutions to be derived here are for arbitrary, but constant Schmidt number and Prandtl number. The cases where Sc and Pr are unity merit special mention because of the simplifications that occur.

If the Schmidt number is 1, the species continuity equation, (2), the momentum equation, (3), and the boundary conditions, (6), require that

$$\frac{u - u^-}{u^+ - u^-} = \frac{c - c^-}{c^+ - c^-}. \tag{7}$$

Thus the concentration is determined immediately from the velocity. If the energy equation is rewritten in terms of the total enthalpy, $h + u^2/2$, the equation shows that if both Sc and Pr are unity

$$\frac{u - u^-}{u^+ - u^-} = \frac{(h + u^2/2) - (h + u^2/2)^-}{(h + u^2/2)^+ - (h + u^2/2)^-}. \tag{8}$$

For this special case both the concentration and temperature have been related to the velocity and it is only necessary to solve the momentum equation. However, values of Sc and Pr different from unity are of interest and each of the equations (1) through (5) will have to be dealt with.

The system of equations describing the mixing problem, equations (1) through (5), is simplified by introducing the Howarth–Dorodnitsyn variables and the stream function. Thus let

$$Y = \int_0^y \frac{\rho}{\rho^+}\,\mathrm{d}y, \qquad u = \frac{\rho^+}{\rho}\frac{\partial\psi}{\partial y}$$

$$\text{and} \qquad v = \frac{-\rho^+}{\rho}\frac{\partial\psi}{\partial x}. \tag{9}$$

It is also convenient to introduce dimensionless variables, so that

$$u^* = \frac{u}{u^+}, \qquad T^* = \frac{T}{T^+}, \qquad c^* = \frac{c}{c^+},$$

$$\rho^* = \frac{\rho}{\rho^+}, \qquad x^* = x/L, \qquad \mu^* = \frac{\mu}{\mu^+},$$

$$Y^* = Y\sqrt{\left(\frac{\rho^+ u^+}{\mu^+ LA}\right)}$$

$$\text{and} \qquad \psi^* = \psi\sqrt{\left(\frac{\rho^+}{\mu^+ L u^+ A}\right)}.$$

Note that L is a reference length, and A is the constant of proportionality in the assumed relation

$$\frac{\mu}{\mu^+} = A\frac{T}{T^+}. \tag{10}$$

For a discussion of this relation see Chapman [1].

The differential equations and boundary conditions admit solutions in terms of the similarity variable $\eta = Y^*(2x^*)^{-\frac{1}{2}}$, so that with $\psi^* = \sqrt{(2x^*)}\, f(\eta)$ the system of equations becomes

$$f\frac{\mathrm{d}c^*}{\mathrm{d}\eta} + \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\rho^* T^*}{Sc}\frac{\mathrm{d}c^*}{\mathrm{d}\eta}\right) = 0 \tag{11}$$

$$f\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\rho^* T^*\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right) = 0 \tag{12}$$

$$\left(1 + \frac{c_{p_1} - c_{p_2}}{c_{p_2}}c^+ c^*\right) f\frac{\mathrm{d}T^*}{\mathrm{d}\eta} + \frac{u^{+2}}{c_{p_2}T^+}\rho^* T^*\left(\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right)^2 + \left(1 + \frac{c_{p_1} - c_{p_2}}{c_{p_2}}c^+ c^*\right)\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\rho^* T^*}{Pr}\frac{\mathrm{d}T^*}{\mathrm{d}\eta}\right)$$
$$+ \left(\frac{c_{p_1} - c_{p_2}}{c_{p_2}}\right)c^+\left(\frac{1}{Pr} + \frac{1}{Sc}\right)\rho^* T^*\frac{\mathrm{d}T^*}{\mathrm{d}\eta}\frac{\mathrm{d}c^*}{\mathrm{d}\eta} = 0 \tag{13}$$

and

$$\rho^* T^*\left(1 + \frac{R_1 - R_2}{R_2}c^+ c^*\right) = 1 + \frac{R_1 - R_2}{R_2}c^+. \tag{14}$$

The energy equation is decoupled from the system since only the product ρT appears in the species continuity, momentum and state equations.

The system of ordinary differential equations, equation (11) through equation (14), may be solved as follows. If $c^+ \neq 0$ and $c^- \neq 0$ and the streams have nearly equal concentration, then one may define an expansion parameter as $(c^+ - c^-)/(c^+ + c^-)$. If $c^- = 0$ and c^+ is small, c^+ is itself an expansion parameter. The different expansion parameters that may be defined are not crucial in themselves, since a choice between them will be dictated by the physical situation. Of greater significance is the method of solving the perturbed equations whose form will be independent of the parameter chosen. In particular, these expansions will lead one to the problem of solving the Blasius equation for the velocity. For the solution of this equation one may follow Görtler and expand in powers of $(u^+ - u^-)/(u^+ + u^-)$, if this parameter is indeed small, or use the classical method of Blasius.

In this paper attention is focused on the problem for which the boundary conditions are

$$\lim_{y\to\infty} c^* = \lim_{y\to\infty} u^* = \lim_{y\to\infty} T^* = \lim_{y\to\infty} \rho^* = 1$$

and

$$\lim_{y\to-\infty} c^* = \lim_{y\to-\infty} u^* = 0,$$

$$\lim_{y\to-\infty} T^* = \frac{T^-}{T^+}, \qquad \lim_{y\to-\infty} \rho^* = \frac{\rho^-}{\rho^+}. \tag{15}$$

Far below the mixing region the fluid is at rest and is composed only of species 2, i.e. $c^- = u^- = 0$. It should be emphasized that the method of solution to be presented here is not limited to the particular group of conditions set forth in equations (15).

f, c^*, ρ^*T^* and T^* are assumed to be expandable in powers of c^+. The equations and boundary conditions for the initial approximation to equations (11) through (15) are

$$f_o \frac{dc_o^*}{d\eta} + \frac{1}{Sc}\frac{d^2c_o^*}{d\eta^2} = 0 \qquad (16)$$

$$f_o \frac{d^2 f_o}{d\eta^2} + \frac{d^3 f_o}{d\eta^3} = 0 \qquad (17)$$

$$f_o \frac{dT_o^*}{d\eta} + \left(\frac{d^2 f_o}{d\eta^2}\right)^2 \frac{u^{+2}}{c_{p_2}T^+} + \frac{1}{Pr}\frac{d^2T_o^*}{d\eta^2} = 0 \qquad (18)$$

$$(\rho^*T^*)_o = 1 \qquad (19)$$

$$\left.\begin{array}{ll} \lim\limits_{\eta\to\infty}\left(\dfrac{df_o}{d\eta}\right) = 1, & \lim\limits_{\eta\to-\infty}\left(\dfrac{df_o}{d\eta}\right) = 0, \\ \lim\limits_{\eta\to\infty} c_o^* = 1, & \lim\limits_{\eta\to-\infty} c_o^* = 0, \\ \lim\limits_{\eta\to\infty} T_0^* = 1 \quad \text{and} & \lim\limits_{\eta\to-\infty} T_0^* = \dfrac{T^-}{T^+}. \end{array}\right\} \qquad (20)$$

The third boundary condition for f_o may be arbitrarily chosen for convenience. This circumstance is discussed by Crane [6] and will not be repeated here. The essential fact is that the boundary-layer equations and boundary conditions have no knowledge of the small pressure difference that actually exists between the streams and the value of η that identifies the dividing streamline is unknown. A solution $f_o(\eta)$ generates an infinity of solutions $f_o(\eta + a)$, where a is an arbitrary constant. Equation (17), the momentum equation, is now decoupled from the system and may be solved independently of the others. Once f_o has been determined, the species continuity equation, (16), can be solved for c_o^* and also the energy equation, (18), can be solved for T_o^*.

The momentum equation

The method for solving the momentum equation, (17), is as follows. Let $\xi = b\eta$ and put $f_o(\eta) = b\bar{f}_o(\xi)$. The momentum equation is unchanged and b is determined from

$$\lim_{\xi\to\infty} b^2 \frac{d\bar{f}_o}{d\xi} = 1. \qquad (21)$$

The boundary condition at minus infinity is still

$$\lim_{\xi\to-\infty} \frac{d\bar{f}_o}{d\xi} = 0. \qquad (22)$$

The third boundary condition on $\bar{f}_o$ is chosen to be

$$\lim_{\xi\to-\infty} \bar{f}_o(\xi) = -1. \qquad (23)$$

The motivation for introducing b and $\bar{f}_o$ derives from the fact that boundary conditions are imposed on f_o both at plus and minus infinity. In order to avoid starting a solution at minus infinity, another at plus infinity and joining them at some finite value of the coordinate ξ, one computes $\bar{f}_o$ somewhat arbitrarily at minus infinity and continues it to positive values of ξ large enough such that $d\bar{f}_o/d\xi$ is sensibly constant. This asymptotic value approached by $d\bar{f}_o/d\xi$ determines a simple scale factor (see equation 21) that enables one to obtain $f_o(\eta)$. This is simply the conversion of a two-point boundary value problem to an initial-value problem. Similar circumstances motivate the methods of solution for both the concentration and the temperature. For $\xi \leqslant 0$ the solution is

$$\left.\begin{array}{l} \bar{f}_o = -1 + \sum\limits_{n=1}^{\infty} \lambda_n \exp(n\xi) \\ \text{where} \quad \lambda_1 = 1 \quad \text{and for} \quad n \geqslant 2 \\ \lambda_n n^2(n-1) + \sum\limits_{j=1}^{n-1} \lambda_j j^2 \lambda_{n-j} = 0. \end{array}\right\} \qquad (24)$$

One can demonstrate the convergence of

$$\sum_{n=1}^{\infty} \lambda_n$$

by noting that for $n = 1$ and $n = 2$

$$|\lambda_n| \leqslant (\tfrac{1}{2})^n.3.$$

Assume that for $j = 1, 2, \ldots, n-1$

$$|\lambda_j| \leqslant (\tfrac{1}{2})^j.3$$

then

$$|\lambda_n| \leqslant \frac{1}{n^2(n-1)} 9 (\tfrac{1}{2})^n \frac{n(2n-1)(n-1)}{6}$$

or

$$|\lambda_n| \leqslant 3 (\tfrac{1}{2})^n.$$

This technique is closely related to the one used by Weyl [11] to show that Blasius' power series has a finite radius of convergence. It may be used to demonstrate the convergence of the other series encountered in this analysis, but the details will not be carried out here.

For $\xi \geqslant 0, \bar{f}_o$ is given by

$$\bar{f}_o = \sum_{n=0}^{\infty} \alpha_n \xi^n \tag{25}$$

α_o, α_1 and α_2 are determined from the solution for $\xi \leqslant 0$. For $n \geqslant 0$ the recursion relation is $-(n+3)(n+2)(n+1)\alpha_{n+3}$

$$= \sum_{j=0}^{n} \alpha_{j+2}\alpha_{n-j}(j+2)(j+1). \tag{26}$$

This power series has a finite radius of convergence whose value is near 3·0. Meksyn [12] has shown that the solution of Blasius' equation has singularities (poles) at approximately $-3{\cdot}1$, $-3{\cdot}1 \exp(2\pi i/3)$ and $-3{\cdot}1 \exp(4\pi i/3)$. The location of these singularities suggests that the power-series solution whose center is at the origin, $\xi = 0$, may be extended by a second power series whose center is at $\xi = 2{\cdot}5$ (for example). The recursion relation for the second power series is identical to that of the first power series. The series centered at 2·5 will also have a finite radius of convergence, but it will enable one to compute $\bar{f}_o$ for values of ξ beyond 5·0. The circumstance that the series for $\bar{f}_o$, equation (25), should have a finite radius of convergence is not expected on physical grounds, and one can avoid the difficulties imposed by the singularities in the complex plane by simply redefining the origin of a power-series expansion. Using simple power series solutions, $\bar{f}_o$ may be easily computed from $\xi = 0$ until those values of ξ are reached at which $\bar{f}_o$ is behaving in an asymptotic manner, i.e. $d\bar{f}_o/d\xi$ is sensibly constant. For a detailed consideration of the asymptotic behavior of $\bar{f}_o$, the reader is referred to Lock [3].

Values of f_o, $df_o/d\eta$ and $d^2f_o/d\eta^2$ are given in Table 1. Numerical results have been obtained by summing the relevant series on an IBM 7094 computer. The convergence of the series solutions is quite rapid and the technique of shifting the origin of the power series removes any artificial limitations on such a solution. The

Table 1. The velocity

η	$f_o(\eta)$	$\frac{df_o}{d\eta}$	$\frac{d^2f_o}{d\eta^2}$
−20·0	−0·8757	0·0	0·0
−15·0	−0·8757	0·0	0·0
−10·0	−0·8756	0·0001	0·0001
−9·0	−0·8754	0·0003	0·0003
−8·0	−0·8750	0·0007	0·0006
−7·0	−0·8738	0·0017	0·0015
−6·0	−0·8712	0·0040	0·0035
−5·0	−0·8648	0·0096	0·0083
−4·6	−0·8602	0·0135	0·0117
−4·2	−0·8538	0·0191	0·0165
−3·8	−0·8446	0·0270	0·0232
−3·4	−0·8317	0·0381	0·0325
−3·0	−0·8136	0·0535	0·0452
−2·6	−0·7881	0·0748	0·0623
−2·2	−0·7527	0·1040	0·0848
−1·8	−0·7036	0·1435	0·1135
−1·4	−0·6362	0·1957	0·1485
−1·0	−0·5451	0·2629	0·1882
−0·6	−0·4237	0·3464	0·2287
−0·2	−0·2659	0·4450	0·2629
0·0	−0·1715	0·4989	0·2747
0·2	−0·0662	0·5546	0·2813
0·6	+0·1781	0·6668	0·2755
1·0	0·4462	0·7713	0·2425
1·4	0·7928	0·8581	0·1888
1·8	1·1495	0·9214	0·1281
2·2	1·5268	0·9616	0·0751
2·6	1·9164	0·9836	0·0377
3·0	2·3121	0·9939	0·0162
3·4	2·7089	0·9980	0·0059
3·8	3·1102	0·9995	0·0019
4·2	3·5101	0·9999	0·0005
4·6	3·9101	1·0000	0·0001
5·0	4·3101	1·0000	0·0000

results show that the power series solution for $\bar{f}_o$ beyond $\xi = 0$ easily extends to those values of ξ where $d\bar{f}_o/d\xi$ is sensibly constant. Fig. 2 is a plot of f_o and its derivative vs. η. The results also show that $b^2 = 0{\cdot}766936$, the scale factor necessary for transforming from $\bar{f}_o$ to f_o. The solution is in agreement with that of Lock [3].

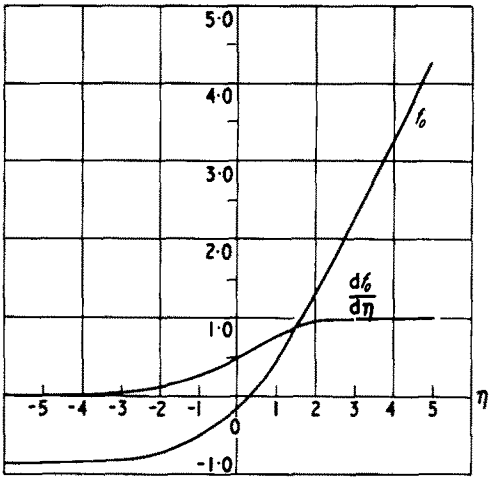

FIG. 2. The velocity.

The concentration equation

The concentration, c_o^*, will be determined by solving

$$Sc\bar{f}_o \frac{d\bar{c}_o}{d\xi} + \frac{d^2\bar{c}_o}{d\xi^2} = 0. \tag{27}$$

At minus infinity we have

$$\lim_{\xi \to -\infty} \bar{c}_o = 0 \tag{28}$$

and a second boundary condition is applied arbitrarily, also at minus infinity. One can be sure however, that $\bar{c}_o$ will asymptotically approach a constant value, $\bar{c}_o(\infty)$, as $\xi \to \infty$ so that the concentration is finally determined by

$$c_o^*(\eta) = \frac{\bar{c}_o(\xi)}{\bar{c}_o(\infty)}. \tag{29}$$

As in the case of the momentum equation, the boundary condition at positive infinity motivates this method of solution. Only a constant of proportionality separates $\bar{c}_o$ from c_o^*. Once this proportionality factor, $\bar{c}_o(\infty)$, has been determined, equation (29) enables one to satisfy the boundary condition on c_o^*, equation (20).

For $\xi \leqslant 0$

$$\bar{c}_o = \exp(\xi Sc) \sum_{n=0}^{\infty} \beta_n \exp(n\xi). \tag{30}$$

β_o is arbitrarily taken to be

$$\beta_o = \frac{1 + Sc}{Sc^2} \tag{31}$$

and β_1 is then equal to $-\lambda_1$. For $n \geqslant 2$ the recursion relation is

$$\beta_n = \frac{-\beta_o Sc^2 \lambda_n}{n(n + Sc)} - \frac{Sc}{n(n + Sc)} \sum_{j=1}^{n-1} (j + Sc)\beta_j \lambda_{n-j}. \tag{32}$$

For $\xi \geqslant 0$, $\bar{c}_o$ is determined from

$$\bar{c}_o = \sum_{n=0}^{\infty} \gamma_n \xi^n \tag{33}$$

where γ_o and γ_1 are determined from the solution for $\xi \leqslant 0$. For $n \geqslant 0$ we have

$$\gamma_{n+2} = \frac{-Sc}{(n+2)(n+1)} \sum_{j=0}^{n} \gamma_{j+1}(j + 1)\alpha_{n-j}. \tag{34}$$

As in the case of $\bar{f}_o$ this power series has a finite radius of convergence also about 3. The value of the radius of convergence is insensitive to the value of Sc, at least for those Schmidt numbers investigated. However, as with $\bar{f}_o$, $\bar{c}_o$ may be computed beyond $\xi = 5{\cdot}0$ by centering a new expansion at $\xi = 2{\cdot}5$. The recursion relation is not altered by shifting the origin of the power series. Values of c_o^* and $dc_o^*/d\eta$ are given in Table 2 with Sc as a parameter. A plot of c_o^* appears in Fig. 3 for various values of Sc. The power series solution satisfactorily computes $\bar{c}_o$ from $\xi = 0$ to those values of ξ at which $\bar{c}_o$ is sensibly constant.

Table 2. The concentration and its derivative

	$Sc = 0{\cdot}5$		$Sc = 1{\cdot}0$		$Sc = 1{\cdot}5$	
η	$c_o^*(\eta)$	$\frac{dc_o^*}{d\eta}$	$c_o^*(\eta)$	$\frac{dc_o^*}{d\eta}$	$c_o^*(\eta)$	$\frac{dc_o^*}{d\eta}$
−20·0	0·0001	0·0000	0·0	0·0000	0·0	0·0000
−15·0	0·0009	0·0004	0·0	0·0000	0·0	0·0000
−10·0	0·0082	0·0036	0·0001	0·0001	0·0	0·0000
−9·0	0·0127	0·0056	0·0003	0·0003	0·0	0·0000
−8·0	0·0197	0·0086	0·0007	0·0006	0·0	0·0000
−7·0	0·0306	0·0134	0·0017	0·0015	0·0001	0·0001
−6·0	0·0474	0·0207	0·0040	0·0035	0·0004	0·0005
−5·0	0·0733	0·0320	0·0096	0·0083	0·0014	0·0018
−4·6	0·0873	0·0380	0·0135	0·0117	0·0023	0·0030
−4·2	0·1182	0·0451	0·0191	0·0165	0·0039	0·0050
−3·8	0·1235	0·0534	0·0270	0·0232	0·0065	0·0084
−3·4	0·1467	0·0632	0·0381	0·0325	0·0109	0·0139
−3·0	0·1742	0·0745	0·0535	0·0452	0·0181	0·0227
−2·6	0·2065	0·0874	0·0748	0·0623	0·0297	0·0367
−2·2	0·2444	0·1020	0·1040	0·0848	0·0485	0·0584
−1·8	0·2883	0·1180	0·1435	0·1135	0·0778	0·0904
−1·4	0·3389	0·1350	0·1957	0·1485	0·1225	0·1353
−1·0	0·3963	0·1520	0·2629	0·1882	0·1878	0·1931
−0·6	0·4603	0·1675	0·3464	0·2287	0·2780	0·2586
−0·2	0·5299	0·1796	0·4450	0·2629	0·3939	0·3187
0·0	0·5662	0·1836	0·4989	0·2747	0·4560	0·3403
0·2	0·6032	0·1858	0·5546	0·2813	0·5295	0·3528
0·6	0·6775	0·1839	0·6668	0·2755	0·6701	0·3420
1·0	0·7491	0·1725	0·7713	0·2425	0·7964	0·2824
1·4	0·8143	0·1522	0·8581	0·1888	0·8921	0·1939
1·8	0·8700	0·1254	0·9214	0·1281	0·9520	0·1084
2·2	0·9143	0·0960	0·9616	0·0751	0·9824	0·0486
2·6	0·9470	0·0680	0·9836	0·0377	0·9947	0·0173
3·0	0·9693	0·0446	0·9939	0·0162	0·9987	0·0049
3·4	0·9834	0·0270	0·9980	0·0059	0·9998	0·0011
3·8	0·9916	0·0151	0·9995	0·0019	1·0000	0·0002
4·2	0·9961	0·0078	0·9999	0·0005	1·0000	0·0000
4·6	0·9983	0·0037	1·0000	0·0001	1·0000	0·0000
5·0	0·9993	0·0016	1·0000	0·0000	1·0000	0·0000
5·4	0·9997	0·0007	1·0000	0·0000	1·0000	0·0000
5·8	0·9999	0·0002	1·0000	0·0000	1·0000	0·0000
6·2	1·0000	0·0001	1·0000	0·0000	1·0000	0·0000
6·6	1·0000	0·0000	1·0000	0·0000	1·0000	0·0000

The temperature equation

The solution for the temperature proceeds in a manner similar to that for the velocity and concentration. The differential equation for T_o is

$$\frac{d^2T_o^*}{d\xi^2} + Pr\bar{f}_o\frac{dT_o^*}{d\xi} + Pr\frac{u^{+2}}{c_{p_2}T^+}b^4\left(\frac{d^2\bar{f}_o}{d\xi^2}\right)^2 = 0 \quad (35)$$

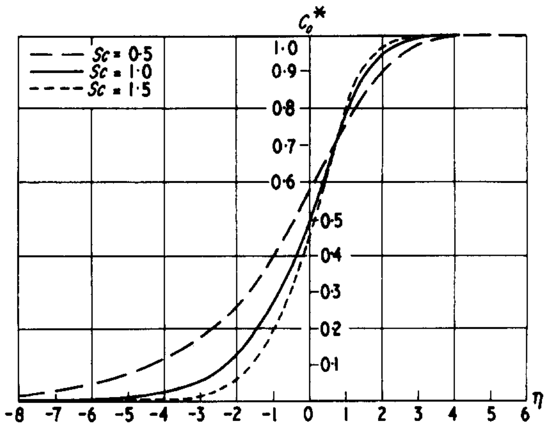

FIG. 3. The concentration.

$$\bar{T}_o = E\bar{T}_{op} + K\bar{T}_{oh} \quad (39)$$

and the boundary conditions are

$$\lim_{\xi\to\infty} T_o^* = 1 \qquad \lim_{\xi\to\infty} T_o^* = \frac{T^-}{T^+}. \quad (36)$$

The homogeneous part of equation (35) is essentially identical to the equation for $\bar{c}_o$, but the temperature equation is distinguished by the presence of an inhomogeneous term. Let

$$T_o^* = \frac{T^-}{T^+}\{\bar{T}_o + 1\}$$

so that the differential equation and boundary conditions satisfied by $\bar{T}_o$ are

$$\frac{d^2\bar{T}_o}{d\xi^2} + Pr\bar{f}_o\frac{d\bar{T}_o}{d\xi} + Pr\left(\frac{u^{+2}}{c_{p_2}T^+}\frac{T^+}{T^-}b^4\right)\left(\frac{d^2\bar{f}_o}{d\xi^2}\right)^2 = 0 \quad (37)$$

and

$$\lim_{\xi\to-\infty} \bar{T}_o = 0 \qquad \lim_{\xi\to\infty} \bar{T}_o = \frac{T^+}{T^-} - 1. \quad (38)$$

The two-point boundary value problem for T_o^* has been recast so that the solution may be started at minus infinity without specifying a value for the parameter T^-/T^+ Denote

$$\frac{u^{+2}}{c_{p_2}T^+}\frac{T^+}{T^-}b^4$$

by E. For a given value of Pr and $E = 1$, we generate a particular solution, $\bar{T}_{op}$, that satisfies the boundary condition at minus infinity. We also compute a solution of the homogeneous equation, $\bar{T}_{oh}$, which depends only on Pr and also satisfies the boundary condition at minus infinity. Then $\bar{T}_o$ is given by

$$\bar{T}_o = E\bar{T}_{op} + K\bar{T}_{oh} \tag{39}$$

where K is determined from

$$\frac{T^+}{T^-} - 1 = E\bar{T}_{op}(\infty) + K\bar{T}_{oh}(\infty). \tag{40}$$

The nondimensional temperature $T_o^*(\eta)$ and its derivative $\mathrm{d}T_o^*/\mathrm{d}\eta$ can now be obtained from the transformations

$$T_o^*(\eta) = \frac{T^-}{T^+}\{K\bar{T}_{oh}(b\eta) + E\bar{T}_{op}(b\eta) + 1\} \tag{41}$$

and

$$\frac{\mathrm{d}T_o^*(\eta)}{\mathrm{d}\eta} = \frac{T^-}{T^+}\left\{Kb\frac{\mathrm{d}\bar{T}_{oh}(b\eta)}{\mathrm{d}(b\eta)} + Eb\frac{\mathrm{d}\bar{T}_{op}(b\eta)}{\mathrm{d}(b\eta)}\right\}. \tag{42}$$

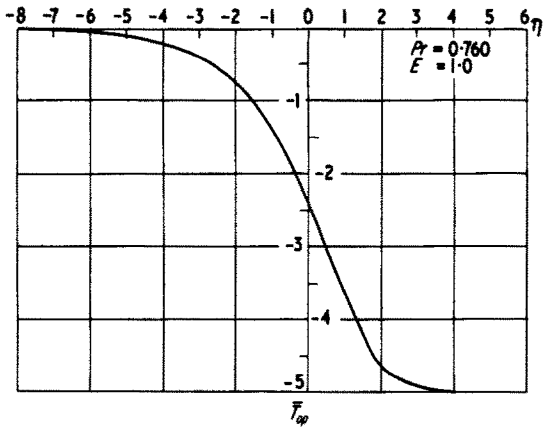

FIG. 4. The temperature.

The existence of a boundary condition at plus infinity has again determined the method of solution. This boundary condition need only be taken into account in the last step of determining K, equation (40). Neither the temperature ratio T^+/T^- nor the parameter E enter into the computations for $\bar{T}_{op}$ and $\bar{T}_{oh}$.

The homogeneous equation for $\bar{T}_o$ is similar to the equation for $\bar{c}_o$, so that only the particular solution requires additional discussion. For $\xi \leqslant 0$ the solution for $\bar{T}_{op}$ is

$$\sum_{n=1}^{\infty} \tau_n \exp(n\xi) + \exp(\xi Pr) \sum_{n=0}^{\infty} \sigma_n \exp(n\xi) \tag{43}$$

where

$$\tau_1 = 0$$

$$\sigma_o = -\frac{1 + Pr}{Pr^2} \tag{44}$$

$$\sigma_1 = +\lambda_1$$

and for $n \geqslant 2$

$$\sigma_n = \frac{-Pr^2\sigma_o\lambda_n}{n(n+Pr)} - \frac{Pr}{n(n+Pr)} \sum_{j=1}^{n-1} \sigma_j\lambda_{n-j}(j + Pr) \tag{45}$$

$$\tau_n = \frac{-Pr}{n(n-Pr)} \sum_{j=1}^{n-1} j\lambda_{n-j}\{\tau_j + j\lambda_j(n-j)^2\}.$$

This solution is valid only if Pr is not an integer. For the case $Pr = 1$, $\bar{T}_{op}$ is given by

$$\sum_{n=1}^{\infty} \tau_n' \exp(n\xi) \tag{46}$$

where τ_1' is arbitrary and for $n \geqslant 2$

$$\tau_n' = \frac{-1}{n(n-1)} \sum_{j=1}^{n-1} j\lambda_{n-j}\{\tau_j' + j\lambda_j(n-j)^2\}. \tag{47}$$

For $\xi \geqslant 0$ the power series solution for $\bar{T}_{op}$ is appropriate and the solution is

$$\sum_{n=0}^{\infty} \theta_n \xi^n \tag{48}$$

where θ_o and θ_1 are determined from the solution for $\xi \leqslant 0$ and for $n \geqslant 0$

$$\theta_{n+2} = \frac{-Pr}{(n+2)(n+1)} \sum_{j=0}^{n} \{(j+1)\theta_{j+1}\alpha_{n-j} + (j+1)(j+2)(n-j+1)(n-j+2) \times \alpha_{j+2}\alpha_{n-j+2}\}. \quad (49)$$

Once again this power series does not converge beyond ξ equal approximately to 3. As with $\bar{f}_o$ and $\bar{c}_o$, a new power series may be centered at $\xi = 2{\cdot}5$ and values of $\bar{T}_{op}$ computed until the function is sensibly constant.

Table 3 presents the values of $\bar{T}_{oh}(b\eta)$, $\bar{T}_{op}(b\eta)$ and their derivatives with respect to $b\eta$ for $Pr = 0{\cdot}76$. Figure 4 depicts $\bar{T}_{op}$ vs. η for $Pr = 0{\cdot}760$. For a given value of T^+/T^- and the parameter E, the constant K is determined from equation (40) and then the expressions for $T_o^*(\eta)$ and $dT_o^*/d\eta$ can be calculated from equations (41) and (42).

Table 3. The temperature and its derivative, Pr = 0·76

η	$\bar{T}_{op}(b\eta)$	$b\dfrac{d\bar{T}_{op}(b\eta)}{d(b\eta)}$	$\bar{T}_{oh}(b\eta)$	$b\dfrac{d\bar{T}_{oh}(b\eta)}{d(b\eta)}$
−20·0	−0·0000	−0·0000	0·0000	0·0000
−15·0	−0·0001	−0·0001	0·0001	0·0001
−10·0	−0·0039	−0·0026	0·0039	0·0026
−9·0	−0·0076	−0·0051	0·0076	0·0051
−8·0	−0·0148	−0·0099	0·0148	0·0099
−7·0	−0·0289	−0·0192	0·0289	0·0192
−6·0	−0·0561	−0·0372	0·0561	0·0372
−5·0	−0·1089	−0·0722	0·1089	0·0721
−4·6	−0·1419	−0·0938	0·1418	0·0937
−4·2	−0·1848	−0·1219	0·1846	0·1215
−3·8	−0·2405	−0·1580	0·2401	0·1574
−3·4	−0·3126	−0·2043	0·3118	0·2030
−3·0	−0·4056	−0·2633	0·4041	0·2608
−2·6	−0·5253	−0·3376	0·5223	0·3327
−2·2	−0·6781	−0·4299	0·6725	0·4407
−1·8	−0·8719	−0·5423	0·8611	0·5251
−1·4	−1·1147	−0·6748	1·0945	0·6440
−1·0	−1·4140	−0·8240	1·3775	0·7712
−0·6	−1·7748	−0·9797	1·7110	0·8943
−0·2	−2·1960	−1·1220	2·0899	0·9942
0·0	−2·4263	−1·1791	2·2923	1·0279
+0·2	−2·6666	−1·2210	2·5000	1·0468
0·6	−3·1623	−1·2419	2·9180	1·0303
1·0	−3·6464	−1·1605	3·3137	0·9352
1·4	−4·0776	−0·9812	3·6571	0·7730
1·8	−4·4233	−0·7418	3·9274	0·5757
2·2	−4·6705	−0·4983	4·2011	0·3835
2·6	−4·8276	−0·2965	4·2390	0·2273
3·0	−4·9161	−0·1561	4·3068	0·1195
3·4	−4·9602	−0·0729	4·3406	0·0557
3·8	−4·9797	−0·0301	4·3555	0·0230
4·2	−4·9874	−0·0110	4·3614	0·0084
4·6	−4·9901	−0·0036	4·3634	0·0027
5·0	−4·9909	−0·0010	4·3640	0·0008
5·4	−4·9911	−0·0003	4·3642	0·0002
5·8	−4·9912	−0·0001	4·3642	0·0000

CONCLUSIONS

Techniques originally used by Blasius to solve the problem of the boundary layer adjacent to a flat plate, have been successfully applied to the zero-th approximation of a perturbation solution of the laminar mixing of compressible fluid mixtures. Previous authors have made some use of these techniques, but only for the purpose of starting numerical integrations. The many series involved in this analysis are all rapidly convergent and the success of the method, in addition to the results themselves, is considered to be a significant aspect of the investigation. It is suggested that the techniques described in this paper may be used to determine the next approximation in the perturbation scheme. It is also reasonable to expect that these methods would be successful in solving other problems such as mass ablation in a boundary layer, where the differential equations would be the same and only the boundary conditions would be modified.

ACKNOWLEDGEMENT

The author would like to express his appreciation to I. Pelech and J. Bleustein for their many helpful discussions during the course of this work. Thanks are also due to Dr. Pelech for suggesting improvements in the content of this paper and to M. Bucciero for his aid in performing the computations.

REFERENCES

1. D. R. CHAPMAN, Laminar mixing of a compressible fluid, N.A.C.A. Report 958 (1950).
2. M. LESSEN, On the stability of the free laminar boundary layer between parallel streams, N.A.C.A. TN 1929 (August, 1949).
3. R. C. LOCK, The velocity distribution in the laminar boundary layer between parallel streams, *Q. Jl. Mech. Appl. Math.* **4**, 42 (1951).
4. H. GÖRTLER, *Z. Angew. Math. Mech.* **22**, 244 (1942).

5. S. I. PAI, *Viscous Flow Theory—I. Laminar Flow*, Chapter IX. Van Nostrand, New York (1954).
6. L. J. CRANE, The laminar and turbulent mixing of jets of compressible fluid, Part II. The mixing of two semi-infinite streams, *J. Fluid Mech.* **3**, 81 (1957).
7. H. BLASIUS, Grenzschichten in Flüssigkeiten mit kleiner Reibung, *Z. Math. Phys.* **56**, 1 (1908); also N.A.C.A. TM No. 1256.
8. C. TRUESDELL and R. A. TOUPIN, *Encyclopedia of Physics*, edited by S. FLÜGGE, edn. III/I, pp. 469–474, 567–568, 612–614, 634–638. Springer-Verlag, Berlin (1960).
9. J. O. HIRSCHFELDER, C. F. CURTISS and R. B. BYRD, *Molecular Theory of Gases and Liquids*, Chapter 8. John Wiley, New York (1954).
10. J. A. FAY and F. R. RIDDELL, Theory of stagnation-point heat transfer in dissociated air, *J. Aeronaut. Sci.* **25**, 73 (1958).
11. H. WEYL, Concerning the differential equations of some boundary layer problems, *Proc. Natn. Acad. Sci. U.S.A.* **27**, 578 (1941).
12. D. MEKSYN, *C. R. Hebd. Séanc. Acad. Sci., Paris* T248(16), 2286 (1959).

Résumé—Le problème étudié ici est celui du mélange laminaire de deux écoulements parallèles de mélanges de fluides compressibles, à deux constituants, par exemple de l'air et quelque impureté. L'intérêt principal est la description du processus de mèlange et son effet sur la concentration de chaque espèce du fluide. La concentration de l'un des constituants du mélange binaire est supposée faible et une solution du type à perturbation est entreprise en prenant la concentration comme petit paramètre. L'approximation d'ordre zéro est développée en détail. Les nombres de Schmidt et de Prandtl sont des constantes arbitraires, on suppose que la viscosité varie linéairement avec la température, et les deux constituants du mélange fluide obéissent chacun à l'équation d'état des gaz parfaits. On obtient une solution analytique pour l'approximation initiale au moyen de développements en série rapidement convergents. Seuls les nombres de Schmidt et de Prandtl influencent le calcul des solutions fondamentales.

Zusammenfassung—Es wird das Problem der laminaren Vermischung zweier Parallelströme kompressibler Gasgemische behandelt. Jeder der beiden Ströme besteht aus zwei Komponenten, z.B. aus Luft und einem Anteil eines anderen Gases. Das Hauptziel ist es, den Vermischungsprozess und seinen Einfluss auf die Konzentration jeder Komponente zu beschreiben. Die Konzentration einer der beiden Komponenten der biniären Mischung wird klein angenommen und es wird, mit dieser Konzentration als kleinem Parameter, eine Störlösung eingeführt. Die nullte Näherung wird detailliert angegeben. Die Schmidt-Zahl und die Prandtl-Zahl sind willkürliche Konstante. Die Temperaturabhängigkeit der Viskosität wird linear angenommen.

Jede der beiden Komponenten des Gemisches genügt für sich einer vollkommenen Zustandsgleichung. Eine analytische Lösung für die Anfangsnäherung ergibt sich mit Hilfe stark konvergierender Reihen. Für die Berechnung fundamentaler Lösungen sind nur die Schmidt-Zahl und die Prandtl-Zahl von Einfluss.

Аннотация—В статье рассматривается задача ламинарного смешивания двух параллельных потоков сжимаемых жидких смесей. Потоки являются двухкомпонентными, например, воздух с какой-нибудь примесью.

Основной целью работы является описание процесса перемешивания и его влияния на концентрацию каждого компонента, которая принимается небольшой. Задача, где концентрация является малым параметром, решается методом возмущений. Приводится подробное решение для нулевого приближения. Числа Шмидта и Прандтля произвольные постоянные, вязкость линейно изменяется с температурой, а оба компонента в отдельности описываются уравнением состояния идеального газа. С помощью быстро сходящихся рядов получено аналитическое решение для нулевого приближения.

Решение основных уравнений зависит только от чисел Шмидта и Прандтля.